Discrete Multivariate Generalized Pareto Distribution with application to dry spells

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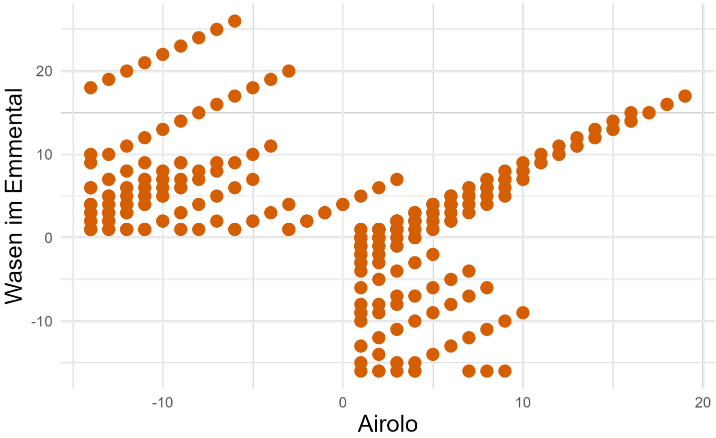
Motivation

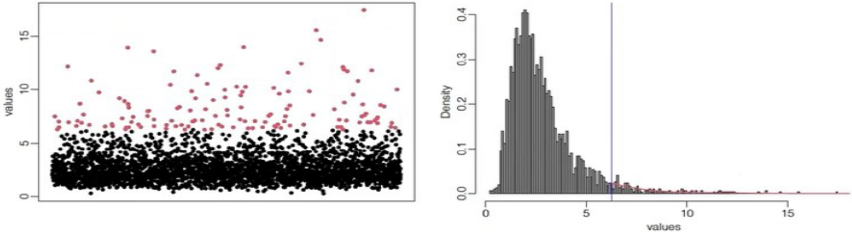
- How to model extremal dependence for multivariate discrete vectors ?
- Examples of interest in real life : insurance claims, number of fires, dry or wet spells.

Dry spells exceedances at two locations in Switzerland



Dry spells over 14 and 16 days

Exceedances in the continuous case



Generalized Pareto Distribution (GPD)

The distribution of exceedances above a large threshold u can be approximated as (Pickands-Balkema-de Haan) :

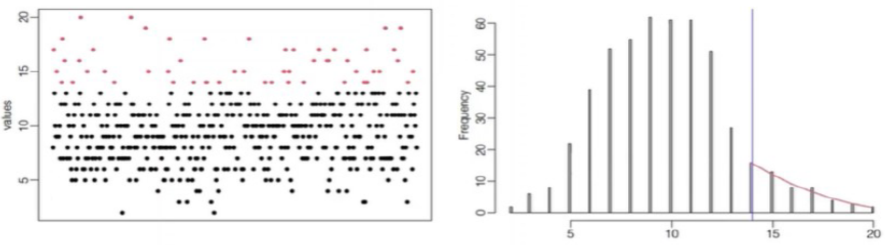
$$\mathbb{P}(Y - u > y \mid Y \geq u) \approx \overline{\text{GPD}}(y; \sigma_u, \xi)$$

where σ_u depends on u, and

$$\overline{\text{GPD}}(y; \sigma_u, \xi) = \left(1 + \xi \frac{y}{\sigma_u}\right)_+^{-\frac{1}{\xi}}, \text{ with } \sigma_u > 0,$$

is the survival function of the GPD.

Discrete GPD



Discrete generalized Pareto distribution (D-GPD) (see [Hitz et al., 2018]).

Discrete GPD definition

From [Hitz et al., 2018], the probability mass function p_{DGPD} of the discrete GPD is defined as, for $k \in \mathbb{N}$,

$$p_{\text{DGPD}}(k; \sigma, \xi) = \overline{\text{GPD}}(k; \sigma, \xi) - \overline{\text{GPD}}(k+1; \sigma, \xi).$$

Mutltivariate GPD

Definition ([Rootzén et al., 2018], Theorem 7)

$\boldsymbol{Z} \in \mathbb{R}^d$ follows a $MGPD(\mathbf{1}, \mathbf{0}, \boldsymbol{S})$ if:

- $\max(\boldsymbol{Z})$ unit exponential distribution,
- $\boldsymbol{S} = \boldsymbol{Z} - \max(\boldsymbol{Z})$ with $\boldsymbol{S}$ independent of $\max(\boldsymbol{Z})$.

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Standard MDGPD

Definition ([Aka et al., 2024])

$\boldsymbol{N} \in \mathbb{Z}^d$ follows a multivariate discrete Generalized Pareto Distribution $MDGPD(\mathbf{1}, \mathbf{0}, \boldsymbol{S})$ if :

- $\max(\boldsymbol{N})$ geometric distribution with parameter $1 - e^{-1}$

$$\mathbb{P}(\max(\boldsymbol{N}) \leq k) = 1 - e^{-k}, k \in \mathbb{N}^*$$

- $\boldsymbol{S} = \boldsymbol{N} - \max(\boldsymbol{N})$ with $\boldsymbol{S}$ independent of $\max(\boldsymbol{N})$.

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Threshold stability

Proposition ([Aka et al., 2024])

$\boldsymbol{N} \sim MDGPD(\mathbf{1}, \mathbf{0}, \boldsymbol{S})$, $\boldsymbol{A} = (a_{ij})$ *a matrix* $\in \mathbb{N}^{n \times d}$ *such that* $\mathbb{P}(\sum_{j=1}^{d} a_{ij} N_j > 0) > 0, \forall i = 1, \dots, n$, *and* $\boldsymbol{m} \in \mathbb{N}^n$, *then*

$$\mathcal{L}(\boldsymbol{AN} - \boldsymbol{m} | \boldsymbol{AN} \not\leq \boldsymbol{m}) = MDGPD(\mathbf{A1}, \mathbf{0}, \boldsymbol{S}_{\boldsymbol{m}}).$$

Generalization of the *MDGPD* by ratio on a Gamma

Proposition ([Aka et al., 2024])

$\boldsymbol{Z}$ MGPD$(\mathbf{1}, \mathbf{0}, \boldsymbol{S})$ and Λ a Gamma(α, β) random variable independent from $\boldsymbol{Z}$. Then,

$$\left\lfloor \frac{\boldsymbol{Z}}{\Lambda} \right\rfloor \text{ follows a MDGPD} \left(\frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}}, \frac{\mathbf{1}}{\boldsymbol{\alpha}}, \boldsymbol{S} \right).$$

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Building **N** from the increments of a distribution

If $\boldsymbol{N} \sim MDGPD(\mathbf{1}, \mathbf{0}, \boldsymbol{S})$, then :

$$\boldsymbol{N} = \boldsymbol{T} - \max(\boldsymbol{T}) + \max(\boldsymbol{N})$$

with $\boldsymbol{T}$ discrete random vector.
In the bivariate case,

$$N_1 = \max(\boldsymbol{N}) + (T_1 - T_2)\mathbb{1}_{((T_1 - T_2) < 0)},$$
$$N_2 = \max(\boldsymbol{N}) - (T_1 - T_2)\mathbb{1}_{((T_1 - T_2) \geq 0)}.$$

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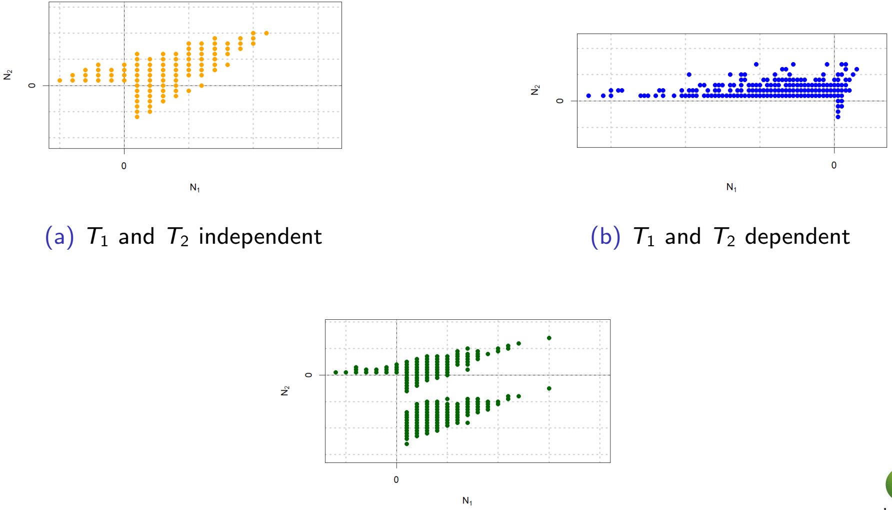
Simulate *MDGPD* from known T

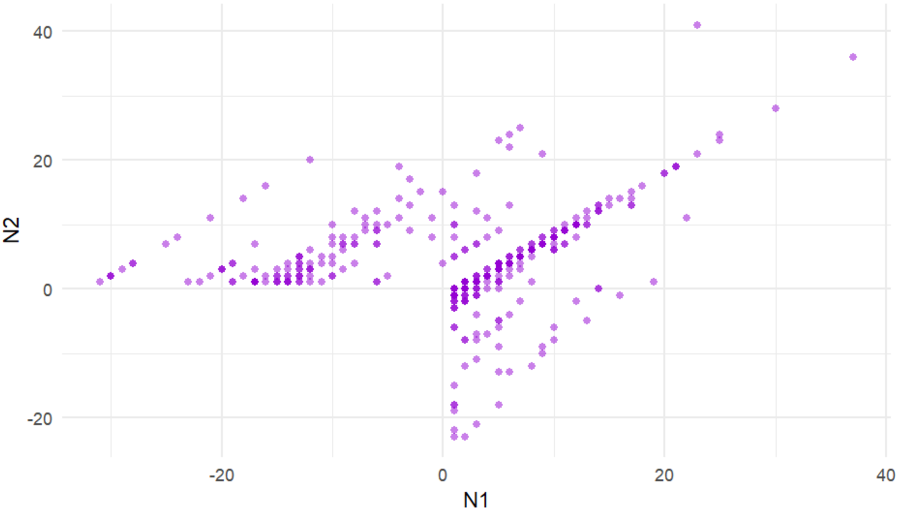


(a) T_1 and T_2 independent

(b) T_1 and T_2 dependent

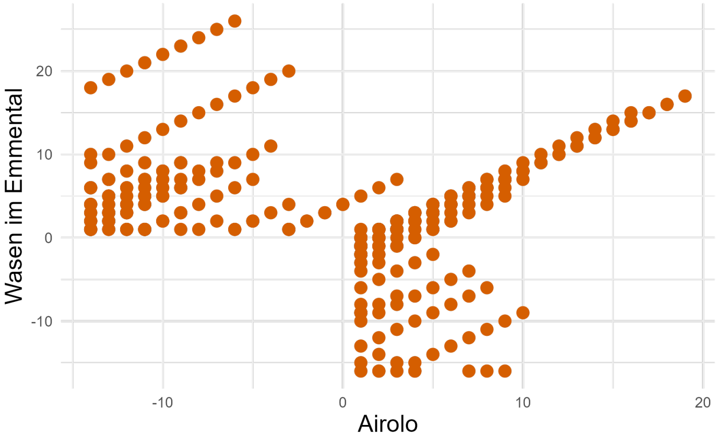
(c) T_1 and T_2 Bimodal

Simulate *MDGPD* from unknown *T*



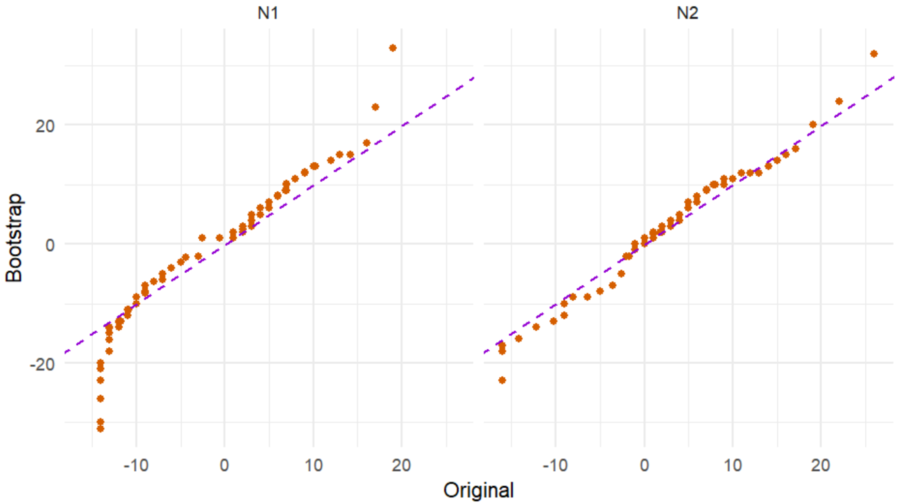
Realization of $\boldsymbol{N}$ using dry spells data

Dry spells exceedances at two locations in Switzerland



Dry spells over 14 and 16 days

Quantile-Quantile plot of marginals



Quantile-Quantile plot of marginals

Future directions

- Fit the *MDGPD* on dry spells for drought assessments,
- Perform regressions with *MDGPD*.

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